Exercise 42

The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z \rangle$ is $\mathbf{F}(\mathbf{r}) = K\mathbf{r}/|\mathbf{r}|^3$ where K is a constant. (See Example 16.1.5.) Find the work done as the particle moves along a straight line from $(2, 0, 0)$ to $(2, 1, 5)$.

Solution

Parameterize the path that the particle moves on by $x = 2$, $y = t$, $z = 5t$ so that

$$\mathbf{r}(t) = \langle 2, t, 5t \rangle, \quad 0 \le t \le 1.$$

Calculate the line integral of the force field over the linear path to find the work done.

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^1 K \frac{\mathbf{r}(t)}{|\mathbf{r}(t)|^3} \cdot \mathbf{r}'(t)\, dt \\ &= K \int_0^1 \frac{\langle 2, t, 5t \rangle}{\left[\sqrt{2^2 + t^2 + (5t)^2}\right]^3} \cdot \langle 0, 1, 5 \rangle\, dt \\ &= K \int_0^1 \frac{(2)(0) + (t)(1) + (5t)(5)}{\left(\sqrt{4 + 26t^2}\right)^3}\, dt \\ &= K \int_0^1 \frac{26t}{(4 + 26t^2)^{3/2}}\, dt \end{aligned}$$

Make the following substitution.

$$\begin{aligned} u &= 4 + 26t^2 \\ du &= 52t\, dt \quad \rightarrow \quad \frac{du}{2} = 26t\, dt \end{aligned}$$

Consequently,

$$\begin{aligned} W &= K \int_{4+26(0)^2}^{4+26(1)^2} \frac{1}{u^{3/2}} \left(\frac{du}{2}\right) \\ &= \frac{K}{2} \int_4^{30} u^{-3/2}\, du \\ &= \frac{K}{2}(-2u^{-1/2})\Big|_4^{30} \\ &= K\left(\frac{1}{2} - \frac{1}{\sqrt{30}}\right). \end{aligned}$$